15
Duration

15.1 Introduction

Duration is a measure of the sensitivity of the price of a bond to changes in the interest rate at which the bond is discounted. It is widely used as a risk measure for bonds (i.e., the higher a bond's duration, the more risky it is). In this chapter we consider a basic duration measure-Macaulay duration-which is defined for the case when the term structure is flat. In Chapter 16 we examine the uses of duration in immunization strategies.

Consider a bond with payments C_t, where $t = 1,..., N$. Ordinarily, the first $N - 1$ payments will be interest payments, and C_N will be the sum of the repayment of principal and the last interest payment. If the term structure is flat and the discount rate for all of the payments is r, then the bond's market price today will be

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+r)^t}$$

The Macaulay duration measure (throughout this chapter and the next, when we use the word "duration" we shall always refer to this measure) is defined as

$$D = \frac{1}{P} \sum_{t=1}^{N} \frac{tC_t}{(1+r)^t}$$

In section 15.3 we will consider the meaning of this formula. Before doing so, however, we show how to calculate the duration in Excel.

15.2 Two Examples

Consider two bonds. Bond A has a 10-year maturity and has just been issued. Its face value is $1,000, and it bears the current market interest rate of 7 percent. Bond B was issued 5 years ago, when interest rates were higher. This bond has $1,000 face value and bears a 13 percent coupon rate. When issued, this bond had a 15-year maturity, so its remaining maturity is 10 years. Since the current market rate of interest is 7 percent, Bond B's market price is given by

$$\$1,421.41 = \sum_{t=1}^{10} \frac{\$130}{(1.07)^t} + \frac{\$1,00}{(1.07)^{10}}$$

It is worthwhile calculating the duration of each of the two bonds (just once!) the long way. We set up a table in Excel:

	A	B	C	D	E	F	G
3	BASIC DURATION CALCULATION						
4							
5	YTM	7%					
6							
7	Year	$C_{t,A}$	$C_{t,A}/P_A*(1+YTM)^t$		$C_{t,B}$	$C_{t,B}/P_B*(1+YTM)^t$	
8	1	70	0.0654		130	0.0855	
9	2	70	0.1223		130	0.1598	
10	3	70	0.1714		130	0.2240	
11	4	70	0.2136		130	0.2791	
12	5	70	0.2495		130	0.3260	
13	6	70	0.2799		130	0.3657	
14	7	70	0.3051		130	0.3987	
15	8	70	0.3259		130	0.4258	
16	9	70	0.3427		130	0.4477	
17	10	1070	5.4393		1130	4.0413	
18		Bond price	Duration		Bond price	Duration	
19		$ 1,000	7.5152		$ 1,421	6.7535	
20							
21		=NPV(B5,B8:B17)				=SUM(F8:F17)	
22							

As might be expected, the duration of bond A is longer than that of bond B, since the average payoff of bond A takes longer than that of bond B. To look at this another way, the net present value of bond A's first-year payoff ($70) represents 6.54 percent of the bond's price, whereas the net present value of bond B's first-year payoff ($130) is 8.55 percent of its price. The figures for the second-year payoffs are 6.11 percent and 7.99 percent, respectively. (For the second-year figures, you have to divide the appropriate line of the preceding table by 2, since in the duration formula each payoff is weighted by the period in which it is received.)

15.2.1 Using an Excel Formula

Excel has two duration formulas, **Duration()** and **MDuration(). MDuration,** somewhat inaccurately termed Macaulay duration by Excel, is defined as

$$\text{MDuration} = \frac{\text{Duration}}{\left(1 + \dfrac{\text{Yield to maturity}}{\text{Number of coupon payments per year}}\right)}$$

Both formulas have the same syntax; for example, for **Duration()** the syntax is as follows:

Duration(settlement, maturity, coupon, yield, frequency, basis) where

settlement is the settlement date (i.e., the purchase date) of the bond.

maturity is the bond's maturity date.

coupon is the bond's coupon.

yield is the bond's yield to maturity.

frequency is the number of coupon payments per year.

basis is the "day count basis" (i.e., the number of days in a year). This is a code between 0 and 4:

0 or omitted	US (NASD) 30/360
1	Actual/actual
2	Actual/360
3	Actual/365
4	European 30/360

The **Duration** formula gives the standard Macaulay duration. The **MDuration** formula can be used in calculating the price volatility of the bond (see section 15.3).

Both duration formulas may require a bit of trickery to implement because they demand a date serial number for both the settlement and the maturity; here, for example, is the calculation of the duration of bond A, implemented using the Excel formula:

	C	D	E	F	G
23	7.5152				
24	=DURATION(DATE(1996,12,3),DATE(2006,12,3),7%,B5,1,1)				

Here we have used any date as the settlement date and then added 10 years to this date to get the maturity date. (The last parameter of the duration formula, which gives the basis, is not strictly necessary and could be omitted.)[1]

1. The insertion of serial date formats in the Excel duration formula is often unhandy. Later in this chapter we use VBA to define a simpler duration formula that overcomes this problem; we postpone this topic until we discuss the calculation of bond duration when the payments are unevenly spaced (section 15.5).

15.3 What Does Duration Mean?

In this section we present three different meanings of duration. Each is interesting and important in its own right.

15.3.1 Duration as the Weighted Average of the Bond's Payments

As originally defined by Macaulay (1938), duration is a weighted average of the bond's payments. Rewrite the duration formula as follows:

$$D = \frac{1}{P}\sum_{t=1}^{N} \frac{tC_t}{(1+r)^t} = \sum_{t=1}^{N}\left[\frac{C_t/P}{(1+r)^t}\right]*t$$

Note that the bracketed terms $\left[\frac{C_t/P}{(1+r)^t}\right]$ sum to 1. This observation follows from the definition of the bond price; each of these terms is the proportion of the bond's price represented by the payment at time t. In the duration formula, each of the terms $\left[\frac{C_t/P}{(1+r)^t}\right]$ is multiplied by its time of occurrence: Thus *the duration is the time-weighted average of the bond's discounted payments as a proportion of the bond's price.*

15.3.2 Duration as the Bond's Price Elasticity with Respect to Its Discount Rate

Viewing duration another way-as the bond's price elasticity with respect to its discount rate-explains why the duration measure can be used to measure the bond's price volatility; it also shows why duration is often used as a risk measure for bonds. To derive this interpretation, we take the derivative of the bond's price with respect to the current interest rate:

$$\frac{dP}{dr} = \sum_{t=1}^{N} \frac{-tC_t}{(1+r)^{t+1}}$$

A little algebra shows that

$$\frac{dP}{dr} = \sum_{t=1}^{N} \frac{-tC_t}{(1+r)^{t+1}} = -\frac{DP}{1+r}$$

which transforms into two useful interpretations of duration:

• First, duration can be regarded as the *discount-rate elasticity of the bond price:*

$$\frac{dP/P}{dr/(1+r)} = \frac{\textit{Percent change in bond price}}{\textit{Percent change in discount factor}} = -D$$

• Second, we can use duration to measure the *price volatility* of a bond by rewriting the previous equation as

$$\frac{dP}{P} = -D\frac{dr}{1+r}$$

Let's go back to the examples of the previous section. Suppose that the market interest rate rises by 10 percent, from 7 percent to 7.7 percent. What will happen to the bond prices? The price of bond A will be

$$\$952.39 = \sum_{t=1}^{10} \frac{\$70}{(1.077)^t} + \frac{\$1,070}{(1.077)^{10}}$$

A similar calculation shows the price of bond B to be

$$1,360.50 = \sum_{t=1}^{10} \frac{\$130}{(1.077)^t} + \frac{\$1,000}{(1.077)^{10}}$$

As predicted by the price-volatility formula, the changes in the bond prices are approximated by D*P* @ -*DP*D*r*/(1 + *r)*. To see this relationship, work out the numbers for each bond:

	H	I	J	K	L	M
5	Approximating Price Changes Using Duration					
6						
7		Actual				
8	Bond	DP	D	P	Dr	-DPD/(1+r)
9	A	-47.61	7.5152	$ 1,000	0.007	-49.17
10	B	-60.92	6.7535	$ 1,421	0.007	-62.80

Note that instead of using the Excel **Duration** function and multiplying by $Dr/(1 + r)$, we could have used the **MDuration** function and multiplied by Dr.

15.3.3 Babcock's Formula: Duration as the Convex Combination of Bond Yields

A third interpretation of duration is Babcock's (1985) formula, which shows that duration is a weighted average of two factors:

$$D = N\left(1 - \frac{y}{r}\right) + \frac{y}{r} PVIF(r, N) * (1 + r)$$

where the "current yield" of the bond is

$$y = \frac{Bond\ coupon}{Bond\ price}$$

and the present value of an N-period annuity is

$$PVIF(r, N) = \sum_{i=1}^{N} \frac{1}{(1 + r)^i}$$

This formula gives two useful insights into the duration measure:

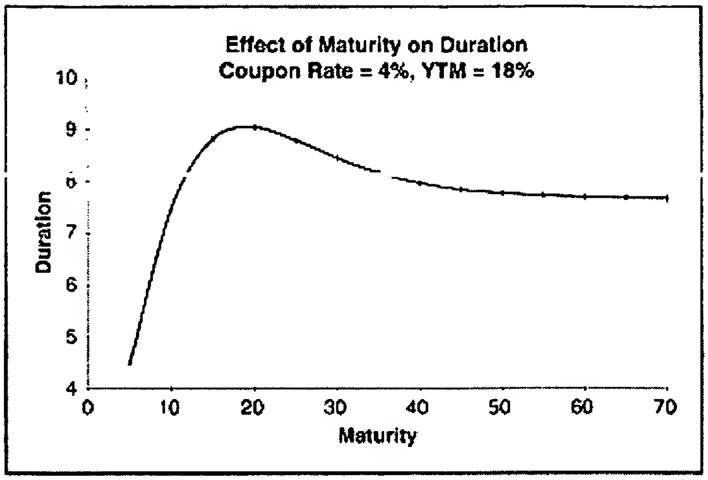
- Duration is a weighted average of the maturity of the bond and of $(1 + r)$ times the PVIF associated with the bond. (Note that the PVIF is given by the Excel formula **PV(r,N,1)**.)

- In many cases the current yield of the bond y is not greatly different from its yield to maturity r. In these cases, duration is not very different from $(1 + r)$ *PVIF*.

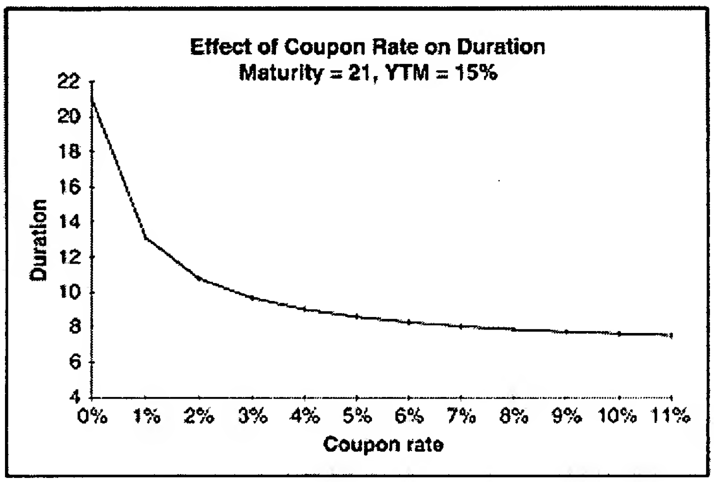
Unlike the two previous interpretations, Babcock's formula holds only for the case of a bond with constant coupon payments and single repayment of principal at time N; that is, the formula does not extend to the case where the payments C_t differ over time.

15.4 Duration Patterns

Intuitively we would expect that duration is an increasing function of a bond's maturity and a decreasing function of a bond's coupon. However, as the following graphs show, this expectation is not always true.



As the coupon increases, the bond's duration unequivocally decreases. The following graph gives an example:



15.5 The Duration of a Bond with Uneven Payments

The duration formulas that we have discussed assume that bond payments are evenly spaced. This is almost invariably the case for bonds, *except for the first payment.* For example, consider a bond that pays interest on May 1 of each of the years 1997, 1998,..., 2010, with repayment of its face value on the last date. All the payments are spaced one year apart; however, if this bond is purchased on

September 1, 1996, then the time to the first payment is eight months, not one year. We shall refer to such a bond as a *bond with uneven payments.* In this section we discuss two problems related to this (extremely common) problem:

• The calculation of the duration of such a bond, when the yield to maturity (YTM) is known. We show that the duration has a very simple formula, related to the duration of a bond with even payments (i.e., the standard duration formula). In the process of the discussion we develop a simpler duration formula in Excel.

• The calculation of the YTM of a bond with uneven payments. This requires a bit of trickery, and ultimately it leads us to another VBA function.

15.5.1 Duration of a Bond with Uneven Payments

Consider a bond with N payments C_a, C_{a+1}, $C_{a+2}, \ldots, C_{a+N-1}$, where $0 < a < 1$. This example is meant to represent a situation where the first payment comes at less than one period from today. The price of such a bond is given by

$$P = \sum_{t=1}^{N} \frac{C_{\alpha+t-1}}{(1+r)^{\alpha+t-1}} = (1+r)^{1-\alpha} \sum_{t=1}^{N} \frac{C_{\alpha+t-1}}{(1+r)^{t}}$$

The duration of this bond is given by

$$D = \frac{1}{P} \sum_{t=1}^{N} \frac{(\alpha+t-1)C_{\alpha+t-1}}{(1+r)^{\alpha+t-1}}$$

Rewrite this last expression as follows:

$$\begin{aligned} D &= \frac{1}{P}(1+r)^{1-\alpha}\left[\sum_{t=1}^{N} \frac{tC_{t+\alpha}}{(1+r)^{t}} + \sum_{t=1}^{N} \frac{(\alpha-1)C_{t+\alpha}}{(1+r)^{t}}\right] \\ &= \frac{1}{(1+r)^{1-\alpha}\sum_{t=1}^{N} \frac{C_{t+\alpha}}{(1+r)^{t}}}(1+r)^{1-\alpha}\left[\sum_{t=1}^{N} \frac{tC_{t+\alpha}}{(1+r)^{t}} + (\alpha-1)\sum_{t=1}^{N} \frac{C_{t+\alpha}}{(1+r)^{t}}\right] \\ &= \frac{1}{\sum_{t=1}^{N} \frac{C_{t+\alpha}}{(1+r)^{t}}}\left[\sum_{t=1}^{N} \frac{tC_{t+\alpha}}{(1+r)^{t}}\right] + \alpha - 1 \end{aligned}$$

The meaning of this equation is the following: A bond with *N* payments, the first occurring at time a from today, has duration which is the sum of

- The duration of a bond with *N* payments spaced at even intervals.
- a - 1

We can see the meaning of this equation in the following example.

	A	B	C	D	E	F	G	H	I
4	**Brute-Force Calculation of Duration of Bond with Uneven Periods**								
5									
6	Alpha	0.3							
7	N	5	Number of payments						
8	YTM	6%							
9	Coupon	100				Cell formula:			
10	Face	1,000				(D13*C13)/(1+$B$8)^C13/$D$20			
11									
12			Period	Payment			Period	Payment	
13			0.3	100	0.0242		1	100	0.0807
14			1.3	100	0.0990		2	100	0.1523
15			2.3	100	0.1653		3	100	0.2156
16			3.3	100	0.2237		4	100	0.2712
17			4.3	1,100	3.0249		5	1,100	3.5173
18					3.5371				4.2371
19									
20		Bond price		1,217				1,168	
21		Bond price cell formula:		NPV(B8,D13:D17)*(1+B8)^(1-B6)					
22									
23		Newly-defined VBA formula			3.5371	<-- =DDURATION(B7,B9/B10,B8,B6)			

As noted in section 15.2, the Excel formula **Duration()** is somewhat difficult to use, because of the insertion of the dates. We therefore write a simpler duration formula using VBA; the syntax of this formula is **DDuration(numberPayments, couponRate, YTM, timeToFirstPayment):**

```
Function dduration(numberPayments, couponRate, YTM,
timeToFirstPayment)

  price = 1 / (1 + YTM) ^ numberPayments
  dduration = numberPayments / (1 + YTM) ^ numberPayments
```

```
For Index = 1 To number Payments
  price = couponRate / (1 + YTM) ^ Index + price
Next Index

For Index = 1 To numberPayments
  dduration = couponRate * Index / (1 + YTM) ^ Index +
  dduration
Next Index
  dduration = dduration / price + timeToFirstPayment - 1

End Function
```

Our homemade formula **DDuration** requires only the number of payments on the bond, the coupon rate, and the time to the first payment a. The use of the formula is illustrated in the previous spreadsheet picture, in cell E23.

15.5.2 Calculating the YTM for Uneven Periods

As the preceding discussion shows, the calculation of duration requires us to know the bond's yield to maturity, which is just the internal rate of return of the bond's payments and its initial price. Often the YTM is given, but when it is not, we can run into a problem that requires us to make an adjustment to the Excel **IRR** function. The problem has to do with unevenly spaced bond payments. This section gives a simple example of this problem and shows how a small trick can solve it.

Consider a bond that currently costs $1,123 and that pays a coupon of $89 on January 1 of each of the next five years. On January 1, 2001, the bond will pay $1,089, the sum of its annual coupon and its face value. The current date is October 3, 1996. The problem in finding the YTM of this bond is that while most of the bond payments are spaced one year apart, there is only 0.2466 of a year until the first coupon payment: 0.2466 = (Date(1997, 1, 1) - Date(1996, 10, 3))/365. Thus we wish to use Excel to solve the following equation.

$$-1{,}123 + \sum_{t=0}^{3} \frac{89}{(1+YTM)^{t+0.2466}} + \frac{1{,}089}{(1+YTM)^{4.2466}} = 0$$

To solve this problem, divide through by $(1 + YTM)^{0.2466}$, as follows:

$$\frac{-1{,}123}{(1+YTM)^{1-0.2466}} + \sum_{t=1}^{4} \frac{89}{(1+YTM)^{t}} + \frac{1{,}089}{(1+YTM)^{5}} = 0$$

This equation is easily solved by Excel provided we use a bit of trickery. Write the following spreadsheet:

	A	B	C	D	E	F	G
2	Current date	3-Oct-96					
3	Annual coupon	89	Paid January 1 for each of next 5 years				
4	Maturity date	1-Jan-01					
5	Face value	1,000					
6	Price of bond	1,123					
7							
8	Time to first payment	0.246575	<-- =(B13-B12)/365				
9							
10				Adjusted			
11		Date	Payment	payment			
12		3-Oct-96	-1123	-1123	<-- =C12/(1+C19)^(1-B8)		
13		1-Jan-97	89	89			
14		1-Jan-98	89	89			
15		1-Jan-99	89	89			
16		1-Jan-00	89	89			
17		1-Jan-01	1089	1089			
18							
19		YTM	0.000%	<-- =IF(C20<>0,0,IRR(D12:D17,0))			
20		Marker	9				

We need "marker" to start the iterations, and to make sure that we don't get caught in a loop of errors. When **marker** is not zero, the spreadsheet looks as it does above. However, when we put **marker** equal to zero, the spreadsheet starts to iterate.[2] After a number of iterations (you may have to push the recalculation key F9 to help the spreadsheet along), you will get the following result, showing that 7.304 percent is the YTM of the bond.

2. This statement assumes that you have checked the "Iteration" box on **Tools¦Options¦Calculation.**

	A	B	C	D	E	F	G
2	Current date	3-Oct-96					
3	Annual coupon	89	Paid January 1 for each of next 5 years				
4	Maturity date	1-Jan-01					
5	Face value	1,000					
6	Price of bond	1,123					
7							
8	Time to first payment	0.246575	<-- =(B13-B12)/365				
9							
10				**Adjusted**			
11		**Date**	**Payment**	**payment**			
12		3-Oct-96	-1123	-1064.91	<-- =C12/(1+C19)^(1-B8)		
13		1-Jan-97	89	89			
14		1-Jan-98	89	89			
15		1-Jan-99	89	89			
16		1-Jan-00	89	89			
17		1-Jan-01	1089	1089			
18							
19		YTM	7.034%	<-- =IF(C20<>0,0,IRR(D12:D17,0))			
20		Marker	0				

15.5.3 Using Excel's XIRR Function to Compute the YTM for Uneven Payments

The procedure of the preceding subsection gives some economic (and programming) insight into the problem of calculating a yield to maturity for unevenly spaced cash flows. However, it is quite cumbersome. If you know the dates of the payments, you can automate the procedure by using Excel's XIRR function.

To use this function, you first have to make sure that the Analysis ToolPak is loaded into Excel. Go to **Tools¦Add-Ins.** This brings up the following menu, in which you have to make sure that **Analysis ToolPak** is checked:

You can now use **XIRR**, which returns the internal rate of return for a schedule of cash flows that is not necessarily periodic. To use this function you have to specify the list of cash flows and the list of dates. You can also provide a guess for the IRR, although this may be left out.

Here's our previous example again:

	B	C	D	E	F
5	**Date**	**Payment**			
6	3-Oct-96	-1123			
7	1-Jan-97	89			
8	1-Jan-98	89			
9	1-Jan-99	89			
10	1-Jan-00	89			
11	1-Jan-01	1089			
12					
13	YTM	7.30%	<-- =XIRR(C6:C11,B6:B11,0)		

15.5.4 Calculating the YTM for Uneven Payments Using a VBA Program

If you do not know the payment dates, you can use VBA to calculate the YTM for a series of uneven payments. Our VBA program following is composed of two functions. The first function, **annuityvalue,** calculates the value $\sum_{t=1}^{N} \frac{1}{(1+r)^t}$ The second function, **unevenYTM,** uses the simple bisection technique to calculate the YTM of a series of uneven payments,[3] leaving you to choose the accuracy **epsilon** of the desired result.

3. As discussed in section 13.4 (when we were calculating the implied volatility of an option), the bisection technique works well if we are looking for the zero of a monotonic function (the IRR of the set of bond cash flows considered here being an example).

```
Function annuityvalue(interest, numberPeriods)

 annuityvalue = 0

 For Index = 1 To numberPeriods
  annuityvalue = annuityvalue + 1 / (1 + interest) ^
  Index
 Next Index

End Function

Function unevenYTM(couponRate, faceValue, bondPrice, _
   numberPayments, timeToFirstPayment, epsilon)
 Dim YTM As Double
 high = 1
 low = 0
 While Abs (annuityvalue(YTM, numberPayments) * _
  couponRate * faceValue + faceValue / (1 + YTM) ^_
  numberPayments - bondPrice / (1 + YTM) ^ _
  (1 - timeToFirstPayment)) >= epsilon
 YTM = (high + low) / 2
 If annuityvalue(YTM, numberPayments) * couponRate _
  * faceValue + faceValue / (1 + YTM) ^ _
  numberPayments - bondPrice / (1 + YTM) ^ _
  (1 - timeToFirstPayment) > 0 Then
     low = YTM
  Else
     h igh = YTM
  End If
  Wend
 unevenYTM = (high + low) / 2

End Function
```

15.6 Nonflat Term Structures and Duration

In a general model of the term structure, payments at time t are discounted by rate r_t, so that the value of a bond is given by

$$P = \sum_{t=1}^{N} \frac{C_t}{(1 + r_t)^t}$$

The duration measure discussed in this chapter assumes either a flat term structure (i.e., $r_t = r$ for all t) or a term structure that shifts in a parallel fashion. When the term structure exhibits parallel shifts, we can write the bond price as

$$P = \sum_{t=1}^{N} \frac{C_t}{(1 + r_t + \Delta t)^t}$$

and then derive a measure of duration by taking the derivative with respect to Dt.

A general model of the term structure should explain how the discount rate r_t for time-t payments comes about, and how the rates at time t change. This is a difficult problem, which we do not discuss in this book (with the exception of showing you one widely used term structure model in the following box).[4]

Vasicek's Term Structure Model

One of the most widely used academic term structure models is that of Vasicek.[5] The Vasicek (1977) term structure model makes two basic assumptions about interest rates: (1) The whole term structure depends on the interest rate for the very shortest term to maturity. This interest rate, termed the *spot interest rate,* is denoted by r. (2) The spot rate r is meanreverting. Changes in r can be written as

$$\Delta r = \alpha(\gamma - r)\Delta t + \rho Z\sqrt{\Delta t}$$

where g is the long-term mean spot interest rate, a is the "push" toward the long-run mean spot rate, and r is the instantaneous standard deviation.

4. Hull (1997, Chapter 17) is a good starting point for a fuller discussion of term structure problems.

5. O. Vasicek (1977), "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* 5: 177-188.

Vasicek shows that the present value factor *v(t, s, r)* at time *t* for time *s* > *t* is given by

$$v[t, s, r\, R(\infty)]$$
$$= \exp\left\{\frac{1}{\alpha}[1 - e^{-\alpha(s-t)}][R(\infty) - r] - (s - t)R(\infty) - \frac{\rho^2}{4\alpha^3}[1 - e^{-\alpha(s-t)}]^2\right\},$$
$$t \leq s$$

where *R*(∞) is the long-run interest rate. If we were to price a bond using the Vasicek model, we might specify a, g, r, and *R*(∞) We could then write the price of the bond as a function of the current spot rate *r:*

$$P = \sum_{s=1}^{N} v[0, s, r, R(\infty)]C_t$$

The Vasicek model is easily programmable in Excel, but a full discussion of the model and its properties is beyond the purview of this book.

Does the complexity of the problem mean that the simple duration measure we present in this chapter is useless? Not necessarily. It may be true that the Macaulay duration measure gives a good approximation for changes in bond value as a result of changes in the term structure, even for the case when the term structure itself is relatively complex and not flat.[6] In this section, we explore this possibility, using data from a file **TermStruc.XLS,** which is on the disk that accompanies this book.[7] The file contains monthly information on the term structure of interest rates in the United States for the period 12.1949-2.87 (i.e., December 1949-February 1987). A typical row of this file looks like this:

date	0mo	1mo	2mo	3mo	4mo	5mo	6mo	9mo
12.1946	0.18	0.32	0.42	0.48	0.52	0.55	0.58	0.65

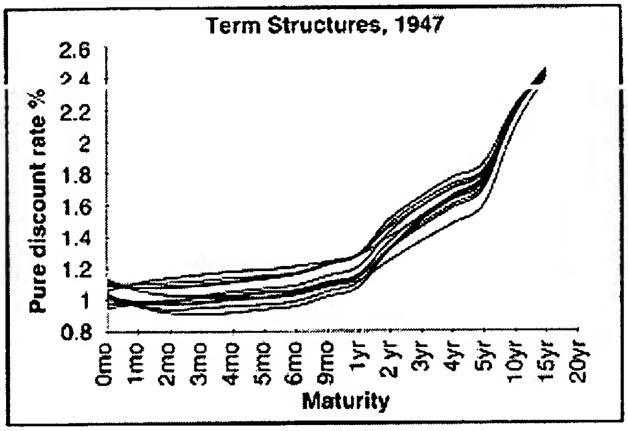
1yr	2yr	3yr	4yr	5yr	10yr	15yr	20yr
0.72	0.95	1.15	1.3	1.41	1.82	2.16	2.32

Interest rates are given in *annual percentage terms;* that is, 0.32 means 0.32 percent per year. Here are some pictures of term structures, taken from the

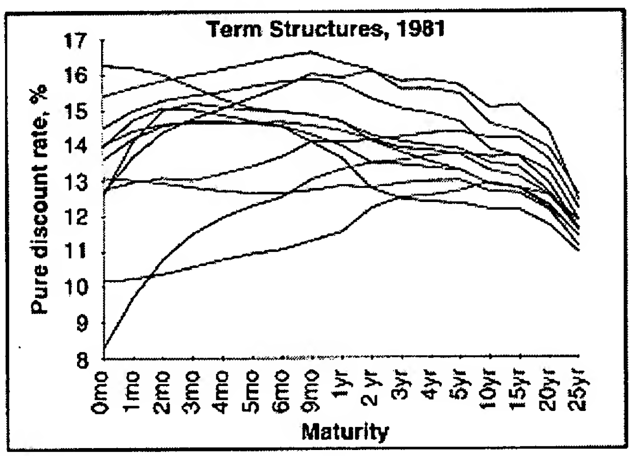
6. A paper by Gultekin and Rogalski (1984) seems to confirm this.

7. The data are from McCullogh (1990).

file.[8] In the following graphs, each line represents the term structure in a particular month. In 1948 the term structures were very closely correlated, and all were upward sloping:



Contrast this graph with the term structures in 1981, when there were upward-and downward-sloping term structures, as well as term structures with "humps":



Despite this great variety of term structure shapes, you will see in exercise 7 that the Macaulay duration can give an adequate approximation to the change in bond value over short periods.

8. The column marked "0mo" gives the *instantaneous interest rate*-the shortest term interest rate in the market. You can think of this as the rate paid by a money market fund on a one-day deposit.

Exercises

1. What is the effect of *raising* the coupon payment on the duration of a bond? Assume that the bond's yield to maturity does not change. Use a numerical example, and plot the answer.

2. What is the effect on a bond's duration of increasing the bond's maturity? As in exercise 1, use a numerical example, and plot the answer. Note that as $N \to \infty$ the bond becomes a consol (a bond that has no repayment of principal but an infinite stream of coupon payments).

The duration of a consol is given by $\frac{1 + YTM}{YTM}$. Show that your numerical answers converge to this formula.

3. "Duration can be viewed as a proxy for the riskiness of a bond. All other things being equal, the riskier of two bonds should have lower duration." Check this claim with an example. What is its economic logic?

4. A pure discount bond with maturity N is a bond with *no payments* at times $t = 1,..., N - 1$; at time $t = N$, a pure discount bond has a single terminal payment of both principal and interest. What is the duration of such a bond?

5. Replicate the two graphs in section 15.4.

6. On January 23, 1987, the market price of a West Jefferson Development Bond was $1,122.32. The bond paid $59 in interest on March 1 and September 1 of each of the years 1987-93. On September 1, 1993, the bond was redeemed at its face value of $1,000. Calculate the yield to maturity of the bond, and then calculate its duration.

7. This exercise relates to the file **TermStruc.XLS.** You are asked to do the following:

a. Produce at least three graphs of six term structures each for 10 typical subperiods. For example, the term structures for January-June 1953, July-December 1980, etc.

b. For January 1980, what would have been the coupon rate on a five-year bond with annual coupons? A 10-year bond? To find these answers you have to solve the following equation:

$$1{,}000 = \sum_{t=1}^{5} \frac{c * 1{,}000}{(1 + r_t)^t} + \frac{1{,}000}{(1 + r_5)^5}$$

where c is the coupon rate on the bond and r_t is the pure discount rate for period t. Note that for a 10-year bond you will have to *interpolate* the data.

c. Calculate the coupon rate on five-year bonds *for all the data.* Graph the results.

d. Now for duration: Return to exercise 7b. Suppose that you have calculated $C_{Jan.80}$ and that immediately following this calculation, the term structure changes to that of February 1980. What will be the effect on the price of the bond? How well is this change approximated by the Macaulay duration measure (assuming that the change in the interest rate Dr is the change in the short-term rate)?

e. Repeat the calculation of exercise 7c for at least 10 periods. Report on the results in an attractive and understandable way.

8. Rewrite the function **DDuration** in section 15.5.1 so that if the **timeToFirstPayment** a is not inserted, then a automatically defaults to 1.

9. Program the Vasicek term structure model in Excel.

16
Immunization Strategies

16.1 Introduction

A bond portfolio's value in the future depends on the interest-rate structure prevailing up to and including the date at which the portfolio is liquidated. If a portfolio has the same payoff at some specific future date, no matter what interest-rate structure prevails, then it is said to be *immunized.* This chapter discusses immunization strategies, which are closely related to the concept of duration discussed in Chapter 15. Immunization strategies have been discussed for many concepts of duration, but this chapter is restricted to the simplest duration concept, that of Macaulay.

16.2 A Basic Simple Model of Immunization

Consider the following situation: A firm has a known future obligation Q. (A good example would be an insurance firm that knows that it has to make a payment in the future.) The discounted value of this obligation is

$$V_0 = \frac{Q}{(1+r)^N}$$

where r is the appropriate discount rate.

Suppose that this future obligation is currently hedged by a bond held by the firm. That is, the firm currently holds a bond whose value V_B is equal to the discounted value of the future obligation V_0. IfP_1, P_2,..., P_M is the stream of anticipated payments made by the bond, then the bond's present value is given by

$$V_B = \sum_{t=1}^{M} \frac{P_t}{(1+r)^t}$$

Now suppose that the underlying interest rate r changes to r + Dr. Using a first-order linear approximation, we find that the new value of the future obligation is given by

$$V_0 + \Delta V_0 \approx V_0 + \frac{dV_0}{dr}\Delta r = V_0 + \Delta r\left[\frac{-NQ}{(1+r)^{N+1}}\right]$$

On the other hand, the new value of the bond is given by

$$V_B + \Delta V_B \approx V_B + \frac{dV_B}{dr}\Delta r = V_B + \Delta r \sum_{t=1}^{N} \frac{-tP_t}{(1+r)^{t+1}}$$

If these two expressions are equal, a change in r will not affect the hedging properties of the company's portfolio. Setting the expressions equal gives us the following condition:

$$V_B + \Delta r \sum_{t=1}^{N} \frac{-tP_t}{(1+r)^{t+1}} = V_0 + \Delta r \left[\frac{-NQ}{(1+r)^{N+1}} \right]$$

Recalling that

$$V_B = V_0 = \frac{Q}{(1+r)^N}$$

we can simplify this expression to get

$$\frac{1}{V_B} \sum_{t=1}^{M} \frac{tP_t}{(1+r)^t} = N$$

This statement is worth restating as a formal proposition:

Suppose that the term structure of interest rates is flat (that is, the discount rate for cash flows occurring at all future times is the same) or that the term structure moves up or down in parallel movements. Then a necessary and sufficient condition that the market value of an asset be equal under all changes of the discount rate r to the market value of a future obligation Q is that the duration of the asset equal the duration of the obligation. Here we understand the word "equal" to mean equal in the sense of a first-order approximation.

An obligation against which an asset of this type is held is said to be *immunized.*

The preceding statement has two critical limitations:

• The immunization discussed applies only to first-order approximations. When we get to a numerical example in the succeeding sections, we shall see that there is a big difference between first-order equality and "true" equality. In *Animal Farm,* George Orwell made the same observation about the barnyard: "All animals are equal, but some animals are more equal than others."

• We have assumed either that the term structure is flat or that the term structure moves up or down in parallel movements. At best, this assumption might be considered a poor approximation of reality (recall the term structure graphs in section 15.6). Alternative theories of the term structure lead to alternative definitions of duration and immunization (for alternatives, see Bierwag et al., 1981, 1983a,b; Cox, Ingersoll, and Ross, 1985; Vasicek, 1977). However, in an

empirical investigation, Gultekin and Rogalski (1984) found that the simple Macaulay duration we use in this chapter works at least as well as any of the alternatives.

16.3 A Numerical Example

In this section we consider a basic numerical immunization example. Suppose you are trying to immunize a 10-year obligation whose present value is $1,000 (this means that, at the current interest rate of 6 percent, its future value is $\$1,000 * (1.06)^{10} = \$1,790.85$. You intend to immunize the obligation by purchasing $1,000 worth of a bond or a combination of bonds. You consider three bonds:

- Bond 1 has 10 years remaining until maturity, a coupon rate of 6.7 perce nt, and a face value of $1,000.
- Bond 2 has 15 years until maturity, a coupon rate of 6.988 percent, and a face value of $1,000.
- Bond 3 has 30 years until maturity, a coupon rate of 5.9 percent, and a face value of $1,000.

At the existing yield to maturity of 6 percent, the prices of the bonds differ. Bond 1, for example is worth

$\$1,051.52 = \sum_{t}^{10} 67/(1.06)^t + 1,000/(1.06)^{10}$ thus, in order to purchase $1,000 worth of this bond, you have to purchase $951 = $1,000/$1,051.52 of *face value* of the bond. Here is a summary for the three bonds:

	A	B	C	D
2	BASIC IMMUNIZATION EXAMPLE WITH 3 BONDS			
3				
4	Yield to maturity	6%		
5				
6		Bond 1	Bond 2	Bond 3
7	Coupon rate	6.70%	6.988%	5.90%
8	Maturity	10	15	30
9	Face value	1,000	1,000	1,000
10				
11	Bond price	$1,051.52	$1,095.96	$986.24
12	Face value equal to $1,000 of market value	$ 951.00	$ 912.44	$ 1,013.96
13				
14	Duration	7.6655	10.0000	14.6361
15				
16		=dduration(B8,B7,$B$4,1)		
17				

Note that to calculate the duration, we have used the "homemade" **DDuration** function defined in Chapter 15.

If the yield to maturity doesn't change, then you will be able to reinvest each coupon at 6 percent. Thus bond 2, for example, will give a terminal wealth at the end of 10 years, of

$$\sum_{t=0}^{9} 69.88 * (1.06)^t + \left[\sum_{t=1}^{5} \frac{69.88}{(1.06)^t} + \frac{1,000}{(1.06)^5}\right] = 921.07 + 1,041.62 = 1,962.69$$

The first term in this expression, $\sum_{t=0}^{9} 69.88 * (1.06)^t$ is the sum of the reinvested coupons. The second and third terms, $\sum_{t=1}^{5} \frac{69.88}{(1.06)^t} + \frac{1,000}{(1.06)^5}$ represent the market value of the bond in year 10 , when the bond has five more years until maturity. Since we will be buying only $912.44 of face value of this bond, we have, at the end of 10 years, 0.91244 * $1,962.69 = $1,790.85. This is exactly the amount we wanted to have at this date. The results of this calculation for all three bonds, provided there is no change in the yield to maturity, are given in the following table:

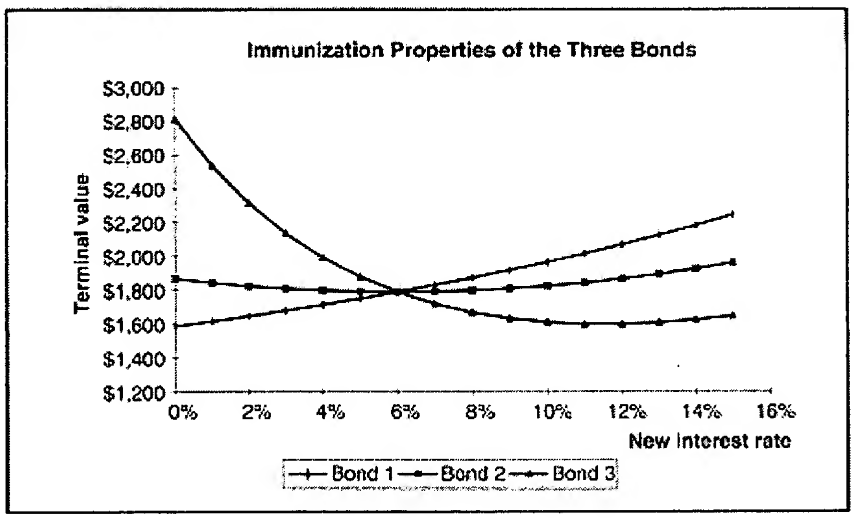
	A	B	C	D	E	F	G	H
21								
22		Bond 1	Bond 2	Bond 3				
23	Bond price	$1,000.00	$1,041.62	$988.53				
24	Reinvested coupons	$883.11	$921.07	$777.67				
25	Total	$1,883.11	$1,962.69	$1,766.20				
26								
27	Multiply by percent of face value bought	95.10%	91.24%	101.40%				
28	Product	$1,790.85	$1,790.85	$1,790.85				
29								
30								
31								
32		=-FV($B$20,10,D7*D9)						
33				=-PV($B$20,D8-10,D7*D9)+D9/(1+$B$20)^(D8-10)				
34								

The upshot of this table is that purchasing $1,000 of any of the three bonds will provide-10 years from now-funding for your future obligation of $1,790.85, *provided the market interest rate of 6 percent doesn't change.*

Now suppose that, immediately after you purchase the bonds, the yield to maturity changes to some new value and stays there. This change will obviously affect the calculations we have already done. For example, if the yield falls to 5 percent, the table will now look as follows.

	A	B	C	D
20	New yield to maturity	5%		
21				
22		**Bond 1**	**Bold 2**	**Bond 3**
23	Bond price	$1,000.00	$1,086.07	$1,112.16
24	Reinvested coupons	$842.72	$878.94	$742.10
25	**Total**	$1,842.72	$1,965.01	$1,854.26
26				
27	Multiply by percent of face value bought	95.10%	91.24%	101.40%
28	**Product**	$1,752.43	$1,792.97	$1,880.14

Thus, if the yield falls, bond 1 will no longer fund our obligation, whereas bond 3 will overfund it. Bond 2's ability to fund the obligation-not surprisingly, in view of the fact that its duration is exactly 10 years-hardly changes. We can repeat this calculation for any new yield to maturity. The results are shown in the following figure.



Clearly, if you want an immunized strategy, you should buy bond 2!

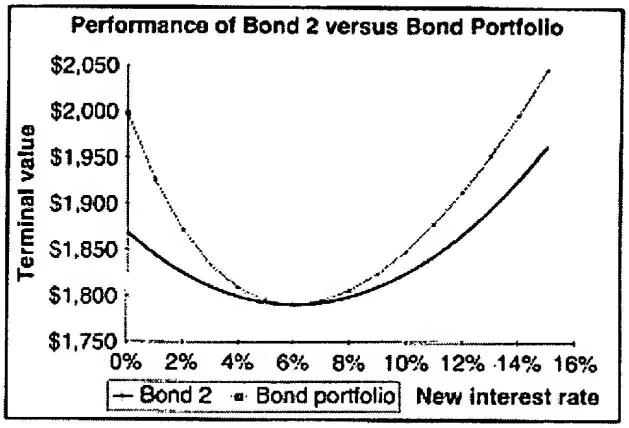
16.4 Convexity: A Continuation of Our Immunization Experiment

The duration of a portfolio is the weighted average duration of the assets in the portfolio. As a result, there is another way to get a bond investment with a duration of 10: If we invest $665.09 in bond 1 and $344.91 in bond 3, the

resulting portfolio also has a duration of 10. These weights are calculated as follows:

$$\lambda * D_{bond\,1} + (1 - \lambda) * D_{bond\,2} = 7.665\lambda + 14.636(1 - \lambda)$$
$$= 10 \Rightarrow \lambda = 0.66509$$

Suppose we repeat our experiment with this portfolio of bonds. As the next figure shows, the portfolio's performance is very respectable; in fact, it is better than that of bond 2 by itself:



Look again at the graph: Notice that, while for both bond 2 and the bond portfolio the terminal value is somewhat convex in the yield to maturity, the terminal value of the portfolio is *more convex* than that of the single bond. Redington (1952), one of the influential propagators of the concept of duration and immunization, thought this convexity very desirable, and we can see why: No matter what the change in the yield to maturity, the portfolio of bonds provides *more overfunding* of the future obligation than the single bond. This is obviously a desirable property for an immunized portfolio, and it leads us to formulate the following rule:

In a comparison between two immunized portfolios, both of which are to fund a known future obligation, the portfolio whose terminal value is more convex with respect to changes in the yield to maturity is preferable.[1]

1. There is another interpretation of the convexity shown in this example: It shows the impossibility of parallel changes in the term structure! If such changes described the uncertainty relating to the term structure, a bond position could be chosen that always benefited from changes in the term structure. Such a position would be an arbitrage, and therefore impossible. I thank Zvi Wiener for pointing this out to me.

16.5 Building a Better Mousetrap

Despite what was said in the preceding section, there is some interest in deriving the characteristics of a bond portfolio whose terminal value is as insensitive to changes in the yield as possible. One way of improving the performance (when so defined) of the bond portfolio is not only to match the first derivatives of the change in value (which, as we saw in section 16.2, leads to the duration concept), but also to match the second derivatives.

A direct extension of the analysis of section 16.2, leads us to the conclusion that matching the second derivatives requires

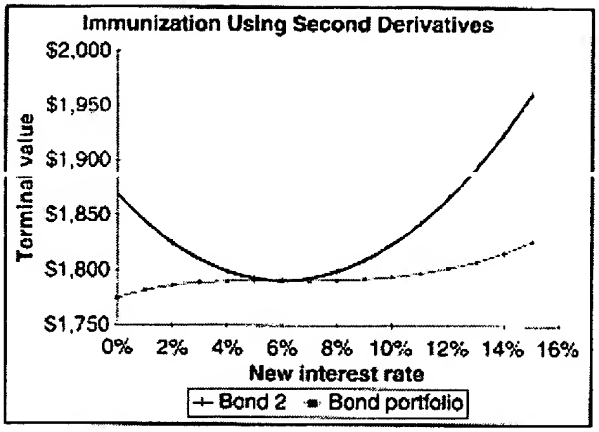
$$N(N + 1) = \sum_{t=1}^{M} \frac{t(t \div 1)P_t}{(1 + r)^t}$$

The following example illustrates the kind of improvement that can be made in a portfolio where the second derivatives are also matched. Consider four bonds, one of which, bond 2, is our old friend from the previous example, whose duration is exactly 10. The bonds are described in the following table:

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**
Coupon rate	4.50%	6.988%	3.50%	11.00%
Maturity	20	15	14	10
Face value	1,000	1,000	1,000	1,000
Bond price	$827.95	$1,095.96	$767.63	$1,368.00
Face value equal to $1,000 of market value	$1,207.80	$912.44	$1,302.72	$730.99
Duration	12.8964	10.0000	10.8484	7.0539
Second duration derivative	229.0873	136.4996	148.7023	67.5980

The last line of the table gives the value of the function $\frac{1}{P}\sum_{t=1}^{N} \frac{t(t + 1)C_t}{(1 + r)^t}$,

which serves as the second parameter for normalizing the portfolio. We need three bonds in order to calculate a portfolio of bonds whose duration and whose second duration derivative are exactly equal to those of the liability. The proportions of a portfolio that sets both the duration and its second derivative equal to those of the liability are as follows: bond 1 = -0.56185, bond 2 = 1.641528, bond 3 = -0.07967. (The negative proportions of bonds 1 and 3 means that we are short-selling these bonds.) As the following figure shows, this portfolio provides a better hedge against the terminal value than even bond 2.



Note that though the portfolio is a better hedge, in the sense that its portfolio varies less and hence immunizes the obligation more thoroughly, bond 2 would be preferable if we were interested in maximizing the convexity of the assets purchased to fund the future obligation.

Exercises

1. Prove that the duration of a portfolio is the weighted average duration of the portfolio assets.

2. Set up a spreadsheet that enables you to duplicate the calculations of section 16.5.

3. An investor is considering investing $100,000 in bond portfolio composed of two bonds:

	Bond A	Bond B
Coupon rate	8%	12%
Maturity	6	25
YTM	9%	9%
Face value	1,000	1,000
Time to first payment	0.6	0.8

a. Calculate the duration of both bonds.

b. How much money should an investor put in Bond A and in Bond B in order to create a bond portfolio with a duration of 9 years?

c. Create a data table showing the relation between the proportion invested in Bond A and the target duration of the portfolio.

17
Calculating Default-Adjusted Expected Bond Returns

17.1 Introduction

In this chapter we discuss the effects of default risk on the returns from holding bonds to maturity. The *expected return* on a bond that may possibly default is different from the bond's *promised return.* The latter is defined as the bond's *yield to maturity*-the internal rate of return calculated from the bond's current market price and its *promised* coupon payments and *promised eventual return* of principal in the future. The bond's expected return is less easily calculated: We need to take into account both the bond's probability of future default and the percentage of its principal which holders can expect to recover in the case of default. To complicate matters still further, default can happen in stages, through the gradual degradation of the issuing company's creditworthiness.[1]

In this chapter we use a Markov model to solve for the expected return on a risky bond. Our adjustment procedure takes into account all three of the factors mentioned: the probability of default, the transition of the issuer from one state of creditworthiness to another, and the percentage recovery of face value when the bond defaults. After illustrating the problem and using Excel to solve a small-scale problem, we use some publicly available statistics to program a fuller spreadsheet model. Finally, we show that this model can be used to derive bond betas, the CAPM's risk measure for securities (discussed in Chapters 5-8).

17.1.1 Some Preliminaries

Before proceeding, we define a number of terms:

• A bond is issued with a given amount of *principal* or *face value.* When the bond matures, the bondholder is promised the return of this principal. If the bond is issued *at par,* then it is sold for the principal amount.

• A bond bears an interest rate called the *coupon rate.* The periodic payment promised to the bondholders is the product of the coupon rate and the bond's face value.

1. Besides default risk, bonds are also subject to term structure risk: The prices of bonds may show significant variations over time as a result of changing term structure. This statement will be especially true for long-term bonds. In this chapter we abstract from term structure risk, confining ourselves only to a discussion of the effects of default risk on bond expected returns.

• At any given moment, a bond will be sold in the market for a *market price*. This price may differ from the bond's face value.[2]

• The bond's *yield to maturity* (YTM) is the internal rate of return of the bond, assuming that it is held to maturity and that it does not default.

American corporate bonds are rated by various agencies on the basis of the bond issuer's ability to make repayment on the bonds. The classification scheme for two of the major rating agencies, Standard and Poor's (S&P) and Moody's, is given in the following table:

Long-Term Senior Debt Ratings

Investment-Grade Ratings			Speculative-Grade Ratings		
S&P	Moody's	Interpretation	S&P	Moody's	Interpretation
AAA	Aaa	Highest quality	BB+	Ba1	Likely to fulfill obligations; ongoing uncertainty
			BB	Ba2	
			BB-	Ba3	
AA+	Aa1	High quality	B+	B1	High-risk obligations
AA	Aa2		B	B2	
AA-	Aa3		B-	B3	
A+	A1	Strong payment capacity	CCC+	Caa	Current vulnerability to default
A	A2		CCC		
A-	A3		CCC-		
BBB+	Baa1	Adequate payment capacity	C	Ca	In bankruptcy or default, or other marked shortcomings
BBB	Baa2		D	D	
BBB-	Baa3				

When a bond defaults, its holders will typically receive some payoff, though less than the promised bond coupon rate and return of principal. We refer to the percent of face value paid off in default as the *recovery percentage*.

17.2 Calculating the Expected Return in a One-Period Framework

The bond's yield to maturity is *not* its expected return. It is clear that both a bond's rating and the anticipated payoff to bond holders in the case of bond

2. Just to complicate matters, in the United States the convention is to add to a bond's listed price the *prorated coupon* between the time of the last coupon payment and the purchase date. The sum of these two is termed the *invoice price* of the bond; the invoice price is the actual cost at any moment to a purchaser of buying the bond. In our discussion in this chapter we use the term *market price* to denote the invoice price.

default should affect its expected return. All other things being equal, we would expect that if two newly issued bonds have the same term to maturity, then the lower-rated bond (having the higher default probability) should have a higher coupon rate. Similarly, we would expect that an issued and traded bond whose rating has been lowered would experience a decrease in price. We might also expect that the lower the anticipated payoff in the case of default is, the lower the bond's expected return will be.

As a simple illustration, we calculate the expected return of a one-year bond that can default at maturity. We use the following symbols:

F = face value of the bond

P = price of bond

Q = annual coupon rate of the bond

p = probability that the bond will *not* default at end of year

l = fraction of bond's value bondholders collect upon default

The bond's expected end-of-year cash flow is p*(1+Q)*F+(1-p)*l*F, and its *expected return* is given by

$$\begin{aligned} \textit{Expected return} &= \frac{\textit{Expected year-end cash flow}}{P} - 1 \\ &= \frac{\pi * (1 + Q) * F + (1 - \pi) * \lambda * F}{P} - 1 \end{aligned}$$

This calculation is illustrated in the following spreadsheet:

	A	B	C	D	E	F
1	**EXPECTED RETURN ON A ONE-YEAR BOND**					
2	**WITH AN ADJUSTMENT FOR DEFAULT PROBABILITY**					
3						
4	Face value, F	100				
5	Price, P	98				
6	Annual coupon rate, Q	16%				
7	Nondefault probability, p	90%				
8	Recovery percentage, l	80%				
9						
10	Expected cash flow	112.4	<--B7*(1+B6)*B4+(1-B7)*B8*B4			
11	Expected return	14.69%	<--=B10/B5-1			

17.3 A Multiperiod, Multistate Markov Chain Problem

We now introduce multiple periods into the problem. In this section we define a basic model using a very simple set of ratings, much simpler than the complex rating system introduced in section 17.1. Section 17.5 will use more realistic data.

We suppose that at any date there are four possible bond "ratings":

A	The highest rating.
B	The next highest rating.
D	The bond is in default for the first time (and hence pays off l of the face value).
E	The bond was in default in the previous period; it therefore pays off 0 in the current period and in any future periods.

The *transitions probability* matrix II is given by

$$\Pi = \begin{bmatrix} \pi_{AA} & \pi_{AB} & \pi_{AD} & 0 \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The probabilities p_{ij} indicate the probability that in *one period* the bond will go from a rating of *i* to a rating of *j* In the numerical examples in section 17.4, we use the following P:

$$\Pi = \begin{bmatrix} 0.99 & 0.01 & 0 & 0 \\ 0.03 & 0.96 & 0.01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What does this matrix P mean?

• If a bond is rated A in the current period, there is a probability of 0.99 that it will still be rated A in the next period. There is a probability 0.01 that it will be rated B in the next period, but it is *impossible* for the bond to be rated A today and E in the subsequent period. While it is possible, in principle, to go from ratings A and B to any of the ratings A, B, D, it is *not* possible to go front A or

B to E. This statement is true because E denotes that default took place in the *previous period.*

• In the example of P, a bond that starts off with a rating of B can-in a subsequent period-be rated A, with a probability of 0.03; be rated B, with a probability of 0.96; or be rated D (and hence in default), with a probability of 0.01.

• A bond that is currently in state D (i.e., first-time default), will necessarily be in E in the next period. Thus the third row of our matrix P will always be [0 0 ... 1].

• Once the rating is in E, it remains there permanently. Therefore, the fourth row of the matr ix P also will always be [0 0 ... 1].

17.3.1 The Multiperiod Transition Matrix

The matrix P defines the transition probabilities over one period. The two-period transition probabilities are given by the matrix product P*P (see discussion of matrix products and array functions in Chapter 23):

$$\begin{aligned}\begin{matrix}\textit{Two-period}\\ \textit{transition probability}\end{matrix} &= \Pi * \Pi \\ &= \begin{bmatrix} 0.99 & 0.01 & 0 & 0 \\ 0.03 & 0.96 & 0.01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0.99 & 0.01 & 0 & 0 \\ 0.03 & 0.96 & 0.01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.9980 & 0.0020 & 0 & 0 \\ 0.0588 & 0.9216 & 0.0096 & 0.0100 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Thus if a bond is rated B today, there is a probability of 5.88 percent that in two periods it will be rated A, a probability of 92.16 percent that in two periods it will be rated B, a probability of 0.96 percent that in two periods it will default (and hence be rated D), and a probability of 1 percent that in two periods it will be rated E. The last rating means, of course, that the bond went into default in the first period.

We can use the **MMult** array function of Excel (see Chapter 23) to calculate multiyear transition probability matrices.

	A	B	C	D
3	**One-period transition matrix**			
4	0.9900	0.0100	0.0000	0.0000
5	0.0300	0.9600	0.0100	0.0000
6	0.0000	0.0000	0.0000	1.0000
7	0.0000	0.0000	0.0000	1.0000
8				
9				
10	**Two-period transition matrix**			
11	0.9804	0.0195	0.0001	0.0000
12	0.0585	0.9219	0.0096	0.0100
13	0.0000	0.0000	0.0000	1.0000
14	0.0000	0.0000	0.0000	1.0000
15				
16	Cells A11:D14 contain array formula			
17	{=MMULT(A4:D7,A4:D7)}			
18				
19	**Three-period transition matrix**			
20	0.9712	0.0285	0.0002	0.0001
21	0.0856	0.8856	0.0092	0.0196
22	0.0000	0.0000	0.0000	1.0000
23	0.0000	0.0000	0.0000	1.0000
24				
25	Cells A20:D23 contain array formula			
26	{=MMULT(A4:D7,A11:D14)}			

In general, the year *t* transition matrix is given by the matrix power P^t. Calculating these matrix powers by the procedure that we have illustrated is cumbersome, so we first define a VBA function that can compute powers of matrices.

```
Function matrixpower (matrix, n)
 If n = 1 Then
  matrixpower = matrix
  Else: matrixpower = Application.Mmult (matrixpower
  (matrix, n - 1), matrix)
 End If
End Function
```

The use of this function is illustrated in the following spreadsheet. The function **Matrixpower** allows a one-step computation of the power of any transition matrix:

	A	B	C	D
1	USING THE FUNCTION MATRIXPOWER			
2				
3	One-period transition matrix			
4	0.9900	0.0100	0.0000	0.0000
5	0.0300	0.9600	0.0100	0.0000
6	0.0000	0.0000	0.0000	1.0000
7	0.0000	0.0000	0.0000	1.0000
8				
9	t	10		
10				
11	0.9159	0.0802	0.0007	0.0032
12	0.2405	0.6754	0.0070	0.0771
13	0.0000	0.0000	0.0000	1.0000
14	0.0000	0.0000	0.0000	1.0000
15				
16				
17		Cells A11:D14 contain array formula		
18		{=MATRIXPOWER(A4:D7,B9)}		
19				

From this example it follows that if a bond started out with an A rating, there is a probability of 0.01 percent that the bond will be in default at the end of ten periods, and there is a probability of 0.03 percent that it will default before the tenth period.

17.3.2 The Bond Payoff Vector

Recall that Q denotes the bond's coupon rate and l denotes the percentage payoff of face value if the bond defaults. The payoff vector of the bond depends on whether the bond is currently in its last period N or whether t

$$Payoff(t) = \begin{cases} \begin{bmatrix} Q \\ Q \\ \lambda \\ 0 \end{bmatrix} & if\ t < N \\ \begin{bmatrix} 1+Q \\ 1+Q \\ \lambda \\ 0 \end{bmatrix} & if\ t = N \end{cases}$$

The first two elements of each vector denote the payoff in nondefaulted states; the third element l is the payoff if the rating is D; and the fourth element 0 is the payoff if the bond rating is E. The distinction between the two vectors depends, of course, on the repayment of principal in the terminal period.

Before we can define the expected payoffs, we need to define one further vector, which will denote the *initial state of the bond.* This current state vector is a vector with a 1 for the current rating of the bond and zeros elsewhere. Thus, for example, if the bond has rating A at date 0, then *Initial* = [1, 0, 0, 0]; if it has date 0 rating of B, *Initial* = [0, 1, 0, 0].

We now define the expected bond payoff in period t:

$$E[Payoff(t)] = Initial * \Pi^t * Payoff(t)$$

17.4 A Numerical Example

We continue using the numerical P from the previous section, and we further suppose that l = 0.8, meaning that a defaulted bond will pay off 80 percent of face value in the first period of default. We consider a bond having the following characteristics:

- The bond is currently rated B.
- Its coupon rate $Q = 8\%$.
- The bond has five more years to maturity.
- The bond's current market price is 98 percent of its face value.

The following spreadsheet shows the facts in the preceding bullet list as well as the payoff vectors of the bond at dates before maturity (in cells F4:F7) and on the maturity date (cells 14:17). The transition matrix is given in cells C10:F13 and the initial vector is given in B15:E15.[3]

The expected bond payoffs are given in cells B19:G19. Before we explain how they were calculated, we note the important economic fact that-if the expected payoffs are as given-then the *bond's expected return* is calculated by **IRR(B19:G19,0).** As Cell B20 shows, this expected return is 7.2447 percent.[4]

	A	B	C	D	E	F	G	H	I	J
1	CALCULATING THE EXPECTED BOND RETURN									
2										
3	Bond price	98.00%				Payoff(t<N)			Payoff(N)	
4	Coupon rate, Q	7%			Cells F4:F7	7%		Cells I4:I7	107%	
5	Recovery rate, λ	80%			are called	7%		are called	107%	
6	Bond term, N	5			"payoff1"	80%		"payoff2"	80%	
7	Initial rating	B			in row 19	0		in row 19	0	
8										
9			A	B	D	E				
10	Transition matrix	A	0.9990	0.0010	0.0000	0.0000				
11		B	0.0300	0.9600	0.0100	0.0000				
12		D	0.0000	0.0000	0.0000	1.0000				
13		E	0.0000	0.0000	0.0000	1.0000				
14										
15	Initial vector	0	1	0	0					
16	=IF(B7="A",1,0)				=IF(B7="B",1,0)					
17										
18	Year	0	1	2	3	4	5	6	7	8
19	Expected payoffs	-0.9800	0.0773	0.0763	0.0754	0.0744	1.0274	0.0000	0.0000	0.0000
20	Expected yield	7.2447%								
21										
22										
23	=IRR(B19:AN19,0)									
24										
25										
26										
27										

```
=IF(C18>bondterm,0,
IF(C18=bondterm,MMULT(initial,MMULT(matrixpower(transition,C18),payoff2)),
MMULT(initial,MMULT(matrixpower(transition,C18),payoff1)
)))
```

3. Note the use of the **IF** statement in translating the bond's initial rating (cell B7) to the initial vector given in B15:E15. To avoid confusion, we might improve this statement by writing it as **IF(Upper(B7)="A",1,0),** etc. This method guarantees that even if the bond's rating is entered as a lowercase letter, the initial vector will come out correctly.

4. The actual formula in cell B20 is **IRR(B19:AN19).** This allows the calculation of the IRR of bonds of maturity up to 40 years.

17.4.1 How to Calculate the Expected Bond Payoffs

As indicated in the previous section, the period *t* expected bond payoff is given by the following formula: *E[payoff(t)] = Initial* * P^t * *Payoff(t).* The formula in row 19 uses two **IF** statements to implement this formula:

=IF(C18>bondterm,0,

IF(C18=bondterm,MMULT(initial,MMULT(matrixpower(transition, C18),payoff2)),

MMULT(initial,MMULT(matrixpower(transition,C18),payoff1)

)))

Here's what this means:

- **First IF:** If the current year is greater than the bond term *N* (in our example *N* = 5), then the payoff on the bond is 0.

- **Second IF:** If the current year is equal to the bond term *N,* then the expected payoff on the bond is **MMULT(initial,MMULT(matrixpower(transition, C18),payoff2)).** Here **transition** is the name for the transition matrix in cells C10:F13 and **payoff2** is the name for the cells I4:I7.

- If the current year *n* is less than the bond term, then the expected payoff on the bond is **MMULT(initial,MMULT(matrixpower(transition,C18),payoff1)),** where **payoff1** is the name for the cells F4:F7.

Copying this formula gives the whole vector of expected bond payoffs.

17.5 Transition Matrices and Recovery Percentages: What Do We Know?

From an extensive survey of bond defaults conducted by Standard & Poor's, it is possible to calculate average rating transition probabilities:

	Rating at End of Year							
Initial Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.9050	0.0859	0.0074	0.0006	0.0011	0.0000	0.0000	0.0000
AA	0.0076	0.9074	0.0762	0.0064	0.0007	0.0014	0.0002	0.0000
A	0.0009	0.0262	0.9069	0.0547	0.0078	0.0028	0.0001	0.0006
BBB	0.0003	0.0027	0.0615	0.8653	0.0536	0.0131	0.0014	0.0020
BB	0.0003	0.0016	0.0070	0.0738	0.8040	0.0924	0.0096	0.0113
B	0.0000	0.0008	0.0034	0.0053	0.0658	0.8384	0.0370	0.0494
CCC	0.0015	0.0000	0.0046	0.0109	0.0163	0.1148	0.6730	0.1790

There is also a considerable amount of data on the recovery rates in bankruptcy from various industries. A table from a recent article by Edward Altman and Velore M. Kishore follows; from this table we can see that the average recovery rate from a variety of industries was 41 percent.

Recovery Rates by Industry: Defaulted Bonds by Three-Digit SIC Code, 1971-95

			Recovery Rate			
Industry	SIC Code	Number of Observations	Average	Weighted Observation	Median Average	Standard Deviation Weighted
Public utilities	490	56	70.47	65.48	79.07	19.46
Chemicals, petroleum, rubber and plastic products	280,290,300	35	62.73	80.39	71.88	27.10
Machinery, instruments, and related products	350,360,380	36	48.74	44.75	47.50	20.13
Services-business and personal	470,632,720,730	14	46.23	50.01	41.50	25.03
Food and kindred products	200	18	45.28	37.40	41.50	21.67
Wholesale and retail trade	500,510,520	12	44.00	48.90	37.32	22.14
Diversified manufacturing	390,998	20	42.29	29.49	33.88	24.98
Casino, hotel, and recreation	770,790	21	40.15	39.74	28.00	25.66
Building materials, metals, and fabricated products	320,330,340	68	38.76	29.64	37.75	22.86
Transportation and transportation equipment	370,410,420,450	52	38.42	41.12	37.13	27.98
Communication, broadcasing, movies, printing, publishing	270,480,780	65	37.08	39.34	34.50	20.79
Financial institutions	600,610,620,630,670	66	35.69	35.44	32.15	25.72
Construction and real estate	150,650	35	35.27	28.58	24.00	28.69
General merchandise stores	530,540,560,570,580,000	89	33.16	29.35	30.00	20.47
Mining and petroleum drilling	100,103	45	33.02	31.83	32.00	18.01
Textile and apparel products	220,230	31	31.66	33.72	31.13	15.24
Wood, paper, and leather products	240,250,260,310	11	29.77	24.30	18.25	24.38

Using the Altman-Kishore and Standard & Poor's data, we can calculate the following spreadsheet:

	A	B	C	D	E	F	G	H	I	J
1	CALCULATING EXPECTED RETURNS USING HISTORIC TRANSITION MATRIX									
2										
3						Payoff(<N)		Payoff(N)		
4	Bond price	99.00%			AAA	11%		AAA	111%	
5	Coupon rate, C	11.00%			AA	11%		AA	111%	
6	Recovery rate, λ	41%			A	11%		A	111%	
7	Bond term, N	5			BBB	11%		BBB	111%	
8	Initial rating	B			BB	11%		BB	111%	
9					B	11%		B	111%	
10					CCC	11%		CCC	111%	
11					D	41%		D	41%	
12					E	0		E	0%	
13										
14	Transition matrix									
15						Rating at end of year				
16	Initial rating	AAA	AA	A	BBB	BB	B	CCC	D	E
17	AAA	0.9050	0.0859	0.0074	0.0006	0.0011	0.0000	0.0000	0.0000	0.0000
18	AA	0.0076	0.9074	0.0762	0.0064	0.0007	0.0014	0.0002	0.0000	0.0000
19	A	0.0009	0.0262	0.9069	0.0547	0.0078	0.0028	0.0001	0.0006	0.0000
20	BBB	0.0003	0.0027	0.0615	0.8653	0.0536	0.0131	0.0014	0.0020	0.0000
21	BB	0.0003	0.0016	0.0070	0.0738	0.8040	0.0924	0.0096	0.0113	0.0000
22	B	0.0000	0.0008	0.0034	0.0053	0.0658	0.8384	0.0370	0.0494	0.0000
23	CCC	0.0015	0.0000	0.0046	0.0109	0.0163	0.1148	0.6730	0.1790	0.0000
24	D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
25	E	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
26										
27										
28		AAA	AA	A	BBB	BB	B	CCC	D	E
29	Initial vector	0	0	0	0	0	1	0	0	0
30										
31										
32										
33	Year	0	1	2	3	4	5	6	7	8
34	Expected payoff	-99.00%	0.12483	0.119207	0.113183	0.1071731	0.872328	0	0	0
35	Expected bond return	7.726%								
36										
37										
38										
39										

=IF(UPPER($B$8)=D28,1,0)

=IRR(B34:L34,0)

=IF(E33>$B$7,0,
IF(E33=$B$7,MMULT($B$29:$J$29,MMULT(matrixpower($B$17:$J$25,E33),$I$4:$I$12)),
MMULT($B$29:$J$29,MMULT(matrixpower($B$17:$J$25,E33),$F$4:$F$12))))

The specific example calculates the expected return on a bond with five more years until maturity, currently rated B, with a coupon rate of 11 percent and a current price of 99 percent of par. The assumption is that the bond's payoff in default will match the Altman-Kishore 41 percent average.[5]

5. The transition matrix given here represents some reworking of publicly available data from Standard & Poor's. S & P, data do not give information on transitions beyond CCC; for purposes of this example, we assume that any transition below CCC is into default D. The reworking of the data was done by the author and not by S&P, and the reworking is for illustrative purposes only.

17.6 Adjusting the Expected Return for Uneven Periods

The spreadsheet of the previous section will calculate the expected bond returns adjusted for default probability and recovery percentage, but it still has one major problem: It assumes that all payments on the bond are evenly spaced; that is, it assumes that there is a full period from the current date to the next coupon payments, two periods to the following coupon, and so on. In many cases, of course, the time to the first coupon payment is less than a full period. As discussed in Chapter 15, there is a simple solution to this problem. We illustrate this solution in the following spreadsheet, which is a modification of the previous spreadsheet:

	A	B	C	D	E	F	G	H	I	J
1	CALCULATING EXPECTED RETURNS USING HISTORIC TRANSITION MATRIX									
2					Adjusted for uneven periods					
3						Payoff(t<N)		Payoff(N)		
4	Bond price	102.00%			AAA	12%		AAA	112%	
5	Coupon rate, C	12%			AA	12%		AA	112%	
6	Recovery rate, λ	55%			A	12%		A	112%	
7	Bond term, N	7			BBB	12%		BBB	112%	
8	Initial rating	B			BB	12%		BB	112%	
9	Time until first payment	0.8			B	12%		B	112%	
10					CCC	12%		CCC	112%	
11					D	55%		D	55%	
12					E	0		E	0%	
13										
14	Transition matrix									
15						Rating at end of year				
16	Initial rating	AAA	AA	A	BBB	BB	B	CCC	D	E
17	AAA	0.9050	0.0859	0.0074	0.0006	0.0011	0.0000	0.0000	0.0000	0.0000
18	AA	0.0076	0.9074	0.0762	0.0064	0.0007	0.0014	0.0002	0.0000	0.0000
19	A	0.0009	0.0262	0.9069	0.0547	0.0078	0.0028	0.0001	0.0006	0.0000
20	BBB	0.0003	0.0027	0.0615	0.8653	0.0536	0.0131	0.0014	0.0020	0.0000
21	BB	0.0003	0.0016	0.0070	0.0738	0.8040	0.0924	0.0096	0.0113	0.0000
22	B	0.0000	0.0008	0.0034	0.0053	0.0658	0.8384	0.0370	0.0494	0.0000
23	CCC	0.0015	0.0000	0.0046	0.0109	0.0163	0.1148	0.6730	0.1790	0.0000
24	D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
25	E	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
26										
27										
28		AAA	AA	A	BBB	BB	B	CCC	D	E
29	Initial	0	0	0	0	0	1	0	0	0
30										
31	=-B4/(1+B35)^(1-B9)				=IF(UPPER($B$8)=D28,1,0)					
32										
33	Year	0	1	2	3	4	5	6	7	8
34	Expected payoff	-100.24%	0.141257	0.135058	0.128264	0.1214055	0.114779	0.108535	0.802214	0
35	Expected bond return	9.092%								
36			=IF(E33>$B$7,0,							
37	=IRR(B34:L34,0)		IF(E33=$B$7,MMULT($B$29:$J$29,MMULT(matrixpower($B$17:$J$25,E33),$I$4:$I$12)),							
38			MMULT($B$29:$J$29,MMULT(matrixpower($B$17:$J$25,E33),$F$4:$F$12)))							
39										

The spreadsheet calculates the expected return of a bond rated B with coupon rate 12 percent, a market price of 102 percent of par, and a recovery percentage of 55 percent. The bond has seven more payments, the last payment being the payment of interest plus principal (the principal here is assumed to be 1).[6] The bond has only 0.8 year until the first payment.

17.7 Computing Bond Betas

A vexatious problem in corporate finance is the computation of bond betas. The model presented in this chapter can be easily used to compute the beta of a bond. Recall that the capital asset pricing model's *security market line* (SML) is given by

$$E(r_d) = r_f + \beta_d[E(r_m) - r_f]$$

where $E(r_d)$ = expected return on debt, r_f = return on riskless debt, and $E(r_m)$ = return on equity market portfolio.

If we know expected return on debt, we can calculate the debt b, provided we know the risk-free rate r_f and the expected rate of return on the market $E(r_m)$. Suppose, for example, that the market risk premium $E(r_m) - r_f$ = 8.4%, and that r_f = 7%. Then a bond having an expected return of 8 percent will have a b of 0.119:

	A	B	C	D
1	**CALCULATING A BOND'S BETA**			
2				
3	Market risk premium, $E(r_m) - r_f$	8.40%		
4	r_f	7%		
5	Expected bond return	8.00%		
6	Implied bond beta	0.119	<-- =(B5-B4)/B3	

6. Note that in all the examples of this chapter, the payments on bonds are assumed to be annual. In point of fact, the payments on many corporate bonds are semiannual, or even quarterly. The adjustment to the spreadsheet is easily made: A bond with a coupon of 11 percent and with semiannual payments will pay 5.5 percent of the face value each half year. Thus all the calculations should be made with 5.5 percent, and a "period" will represent one-half year.

Exercises

1. A newly issued bond with one year to maturity has a price of 100, which equals its face value. The coupon rate on the bond is 15 percent; the probability of default in one year is 35 percent; and the bond's payoff in default will be 65 percent of its face value. Calculate the bond's expected return.

2. Consider a case of five possible rating states: A, B, C, D, and E. The states A, B, and C are initial bond ratings; D symbolizes first-time default; and E indicates default in the previous period. Assume that the transition matrix P is given by

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.06 & 0.90 & 0.03 & 0.01 & 0 \\ 0.02 & 0.05 & 0.88 & 0.05 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A 10-year bond issued today at par with an A rating is assumed to bear a coupon rate of 7 percent.

a. If a bond is issued today at par with a B rating and with a recovery percentage of 50 percent, what should be its coupon rate so that its expected return will also be 7 percent?

b. If a bond is issued today at par with a C rating and with a recovery percentage of 50 percent, what should be its coupon rate so that its expected return will also be 7 percent?

3. A bond of XYZ Corporation has the following characteristics:

Market price: 108.32 percent of par

Coupon rate: 15 percent

Number of annual payments (including return of principal) left on bond: 15

Time to first payment: 8 months

XYZ Corporation's debt is currently rated CCC. Use the model of section 17.5 to calculate the bond's expected return. Assume a recovery percentage $l = 78\%$.

4. An underwriter issues a new seven-year B bond at par with a coupon rate of 9 percent. If the expected rate of return on the bond is 8 percent, what is the bond's implied recovery percentage l? Assume the transition matrix given in section 17.5.

5. An underwriter issues a new seven-year CCC bond. The anticipated recovery rate in default of the bond is expected to be 55 percent. What should be the coupon rate on the bond so that its expected return is 9 percent? Assume the transition matrix given in section 17.5.

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